Fokker-Planck-Kramers equation for a Brownian gas in a magnetic field

J. I. Jiménez-Aquino* and M. Romero-Bastida†

Departamento de Física, Universidad Autónoma Metropolitana, Iztapalapa, Apartado Postal 55–534, C.P. 09340, Mexico, D.F., Mexico

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In this work we give an alternative method to calculate the transition probability densities (TPD) for the velocity space, phase space, and Smoluchowsky configuration space of a Brownian gas of charged particles in the presence of a constant magnetic field. Our proposal consists in transforming, by means of a rotation matrix, the Langevin equation of a charged particle in the velocity space into another velocity space where the behavior is quite similar to that of ordinary Brownian motion. A similar strategy is also applied to the phase-space. In consequence, in the transformed space both the Fokker-Planck and Fokker-Planck-Kramers equations are solved following Chandrasekhar's methodology. Our results are compared with those obtained by Czopnik and Garbaczewski [Phys. Rev. E **63**, 021105 (2001)].

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I. INTRODUCTION

The stochastic diffusion of a plasma across a magnetic field arising from the fluctuations of the electric field was first solved by Taylor in 1961 using a Langevin theoretical description [1]. In the next year, the same problem was solved by Kurşunoğlu [2] by making an extension to Chandrasekhar's [3] treatment of ordinary Brownian motion in terms of an auxiliary matrix-valued function, which is not a genuine transition probability density governing the velocity space and the configuration space. Almost forty years later, the problem has again proved to be of interest to other scientists, as shown in Refs. [4–8]. In particular, in Ref. [4] the full description of the Brownian motion in the magnetic field is given through the transition probability densities for the velocity space, phase space, and the Smoluchowsky configuration space. In this reference, the main results have been obtained both by using some of Chandrasekhar's strategies as well as correlation matrices. On the other hand, in Ref. [6] the fundamental solution of the Fokker-Planck equation for heavy ions in a fluid and under the influence of a time-varying electric field has been proposed by using other methods of solution. In Ref. [7], a general expression for the transition probability densities (TPD) describing the anisotropic diffusion across the magnetic field, produced by the stochastic Langevin force, has been calculated by Holod *et al.* (2005), through the Fokker-Planck formalism. In this work, the case of isotropic diffusion [4,8] across the external magnetic field arises as a particular case.

In this work we give an alternative method of solution for a Brownian gas of charged particles in constant magnetic field. Our proposal is different and simpler than those proposed in the above references. It consists in making a transformation, by means of a rotation matrix, to the Langevin equation for a charged particle in the velocity space $\mathbf{u}$, to another velocity space $\mathbf{u}'$, in which the resulting Langevin equation is quite similar to that of the ordinary Brownian motion. Thus, in this transformed velocity space, it is easy to construct and solve the associated Fokker-Planck equation. By returning to the velocity space $\mathbf{u}$, we obtain the same TPD as that calculated by Czopnik and Garbaczewski [4] by other method. The same strategy is applied to the Langevin equation in the phase space $(\mathbf{r},\mathbf{u})$ in such a way that, in the transformed phase space $(\mathbf{r}',\mathbf{u}')$, the Langevin equation will also be very similar to the ordinary Brownian motion and therefore easier to work with. The TPD obtained in the phase space $(\mathbf{r},\mathbf{u})$ can be decomposed in two independent transition probability densities, one describing the diffusion process on the xy plane and the other along the z axis, which is precisely the TPD for free Brownian motion. The planar TPD is not exactly the same, but nevertheless very similar, to that calculated by Czopnik and Garbaczewski.

The structure of this work is the following: In Sec. II, the Langevin equation and its associated Fokker-Planck equation for several variables are introduced in the manner of Risken [9]. Both equations are also established when a transformation of variables takes place. Because our proposal is related to the free Brownian motion, we briefly introduce the solution of both the Fokker-Planck and Fokker-Planck-Kramers equations for this problem following Chandrasekhar's proposal. Our contribution starts in Sec. III, where we establish the strategy to solve the Fokker-Planck equation of the Brownian gas in the velocity space $\mathbf{u}$. Similar strategy is applied in Sec. IV to solve the Fokker-Planck-Kramers equation for the phase space. From the obtained TPD we calculate the planar TPD in the Smoluchowsky configuration space and in the velocity space. Conclusions are finally given in Sec. V.

II. LANGEVIN AND FOKKER-PLANCK EQUATIONS FOR SEVERAL VARIABLES

In Ref. [9] it has been well established that for N stochastic variables $\{\xi\}=\xi_1,\xi_2,\ldots,\xi_N$ the general Langevin equations have the form $(i,j=1,2,\ldots,N)$

$$\dot{\boldsymbol{\xi}} = h_i(\{\xi\},t) + g_{ij}(\{\xi\},t)\Gamma_j(t), \tag{1}$$

where $h_i(\{\xi\},t)$ and $g_{ij}(\{\xi\},t)$ are in general nonlinear functions of $\{\xi\}$ and t. In the following we will use Einstein's

*Electronic address: ines@xanum.uam.mx

†Electronic address: rbm@xanum.uam.mx

summation convention. The $\Gamma_j(t)$ are Gaussian random variables with zero mean value and correlation functions

$$\langle \Gamma_i(t) \rangle = 0, \quad \langle \Gamma_i(t)\Gamma_j(t') \rangle = 2\delta_{ij}\delta(t-t'). \tag{2}$$

The Fokker-Planck or forward Kolmogorov equation for the transition probability density $P(\{x\},t|\{x'\},t')$ associated with the Langevin equation (1), together with the property (2), reads as $(t \geq t')$

$$\frac{\partial P}{\partial t} = \left(-\frac{\partial}{\partial x_i} D_i(\{x\},t) + \frac{\partial^2}{\partial x_i \, \partial x_j} D_{ij}(\{x\},t) \right) P, \tag{3}$$

where $D_i(\{x\},t)$ is the drift coefficient and $D_{ij}(\{x\},t)$ the diffusion coefficient defined, respectively, as

$$D_i(\{x\},t) = h_i(\{x\},t) + g_{kj}(\{x\},t) \frac{\partial}{\partial x_k} g_{ij}(\{x\},t), \tag{4}$$

$$D_{ij}(\{x\},t) = g_{ik}(\{x\},t) g_{jk}(\{x\},t), \tag{5}$$

with initial condition

$$P(\{x\},t'|\{x'\},t') = \delta(\{x\} - \{x'\}). \tag{6}$$

Here $\xi_k(t)=x_k$ for $k=1,2,\ldots,N$, represents the sharp value of the variable $\xi(t)$ at time t. If we multiply Eq. (3) by $W(\{x'\},t')$ and integrate over x' we obtain the Fokker-Planck equation for the probability density $W(\{x\},t)$, i.e.,

$$\frac{\partial}{\partial t} W = \left(-\frac{\partial}{\partial x_i} D_i(\{x\},t) + \frac{\partial^2}{\partial x_i \, \partial x_j} D_{ij}(\{x\},t) \right) W. \tag{7}$$

The complete information of a Markov process is contained in the joint probability distribution $W_2(\{x\},t;\{x'\},t')$ which can be expressed as

$$W_2(\{x\},t;\{x'\},t') = P(\{x\},t|\{x'\},t') W(\{x'\},t'). \tag{8}$$

If the drift and diffusion coefficients do not depend on time, a stationary solution may exist. In this case, P can depend only on the time difference $t-t'$, and we may write for $t \geq t'$ the joint probability distribution in the stationary state

$$W_2(\{x\},t;\{x'\},t') = P(\{x\},t-t'|\{x'\},0) W_{st}(\{x'\}). \tag{9}$$

A particular case of Eq. (1) is the Ornstein-Uhlenbeck [10,11] process, which is described by

$$\dot{\xi}_i = -\gamma_{ij}\xi_j + \Gamma_j(t), \tag{10}$$

with δ-correlated Gaussian distributed forces

$$\langle \Gamma_j(t) \rangle = 0, \quad \langle \Gamma_i(t)\Gamma_j(t') \rangle = q_{ij}\delta(t-t'), \tag{11}$$

where the coefficients $q_{ij}=q_{ji}$ describing the strength of the noise do not depend on the variables ξ_k and γ_{ij} is a constant matrix. For this type of process, the drift coefficient is linear and the diffusion coefficient constant, that is,

$$D_i = -\gamma_{ij}x_j, \quad D_{ij} = D_{ji}, \tag{12}$$

where the matrix D_{ij} is also constant.

A. Transformation of variables

The Langevin equations are very convenient to calculate the drift and diffusion coefficients if the variable transformation is performed. If we introduce new variables ξ' in Eq. (1) such that

$$\xi' = \xi'(\{\xi\},t), \tag{13}$$

then the Langevin equation in the new variables is

$$\dot{\xi}' = h_i'(\{\xi'\},t) + g_{ij}'(\{\xi'\},t)\Gamma_j(t), \tag{14}$$

where

$$h_i' = \frac{\partial \xi_i'}{\partial t} + \frac{\partial \xi_i'}{\partial \xi_k} h_k; \quad g_{ij}' = \frac{\partial \xi_i'}{\partial \xi_k} g_{kj}. \tag{15}$$

Hence, the transformed drift and diffusion coefficients (writing x_i' instead of ξ' in the argument) are given by

$$D_i' = \left(\frac{\partial x_i'}{\partial t} \right)_x + \frac{\partial x_i'}{\partial x_k} D_k + \frac{\partial^2 x_i'}{\partial x_r \, \partial x_k} D_{rk}, \tag{16}$$

$$D_{ij}' = \frac{\partial x_i'}{\partial x_r} \frac{\partial x_j'}{\partial x_k} D_{rk}, \tag{17}$$

where D_i and D_{ij} are given, respectively, by Eqs. (4) and (5). The Fokker-Planck equation (7) is also transformed in such a way that, for the new variables, it can be written as

$$\left(\frac{\partial W'}{\partial t} \right)_{x'} = \left(-\frac{\partial}{\partial x_i'} D_i' + \frac{\partial^2}{\partial x_i' \, \partial x_j'} D_{ij}' \right) W', \tag{18}$$

where the transformed drift D_i' and diffusion D_{ij}' coefficients agree with Eqs. (16) and (17). The probability density W is transformed as $W'=JW=W/J'$, where J is the Jacobian of the transformation defined by

$$J \equiv d^N x / d^N x' = |\mathrm{Det}(\partial x_i / \partial x_j')| = 1/J' = 1/|\mathrm{Det}(\partial x_i' / \partial x_j)|, \tag{19}$$

and $d^N x$ and $d^N x'$ are the volume elements.

B. The Fokker-Planck equation for free Brownian motion in the velocity space

The three-dimensional Brownian motion for the velocity $\mathbf{u}(t)$ in the absence of external force is the simplest form of Eq. (10) and it is given by

$$\dot{\mathbf{u}} = -\beta \mathbf{u} + \mathbf{A}(t), \tag{20}$$

where $-\beta\mathbf{u}$ is the systematic force representing the dynamical friction. The components of vector $\mathbf{A}(t)$ satisfy the same properties as that of Eq. (11), but

$$\langle A_i(t) A_j(t') \rangle = 2q\delta_{ij}\delta(t-t'), \tag{21}$$

q being the noise intensity given by $q=\beta k_B T/m$. If this parameter is absorbed in the function g_{ij}, then it can be shown that drift and diffusion coefficient read as

$$D_i = -\beta u_i, \quad D_{ij} = q\delta_{ij}. \tag{22}$$

The Fokker-Planck equation for the transition probability density of the velocity conditioned by initial data $\mathbf{u}_0$ at time $t_0=0$, i.e., $P(\mathbf{u},t|\mathbf{u}_0)$, as required by Eq. (3) is

$$\frac{\partial P}{\partial t} = \beta \frac{\partial}{\partial u_i}(u_i P) + q \frac{\partial^2}{\partial u_i^2} P = \beta \,\mathrm{div}_{\mathbf{u}}\, \mathbf{u} P + q \nabla_{\mathbf{u}}^2 P, \quad (23)$$

where the operator $\mathrm{div}_{\mathbf{u}}$ and the Laplace operator $\nabla_{\mathbf{u}}^2$ act with respect to the velocity coordinates. The transition probability satisfies the initial condition

$$\lim_{t\to 0} P(\mathbf{u},t|\mathbf{u}_0) \equiv \delta^3(\mathbf{u}-\mathbf{u}_0) = \delta(u_1-u_{01})\,\delta(u_2-u_{02})\,\delta(u_3-u_{03}). \quad (24)$$

Following Chandrasekhar's proposal [3], the general solution of Eq. (23) is connected with its associated first-order equation, i.e.,

$$\frac{\partial P}{\partial t} - \beta \,\mathrm{div}_{\mathbf{u}}\, \mathbf{u} P = 0. \quad (25)$$

The general solution of this first-order equation involves the first three integrals of the Lagrangian subsidiary system

$$\dot{\mathbf{u}} = -\beta \mathbf{u}. \quad (26)$$

The required integrals are therefore

$$\mathbf{u} e^{\beta t} = \mathbf{u}_0 = \mathbf{I_1} = \text{const}. \quad (27)$$

So, the solution of Eq. (23) is then given by

$$P(\mathbf{u},t|\mathbf{u}_0) = \frac{1}{[2\pi q(1-e^{-2\beta t})/\beta]^{3/2}} \exp\left(-\frac{\beta|\mathbf{u}-\mathbf{u}_0 e^{-\beta t}|^2}{2q(1-e^{-2\beta t})}\right). \quad (28)$$

As the time $t\to\infty$ this probability density becomes the Maxwellian distribution, i.e.,

$$P(\mathbf{u}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m|\mathbf{u}|^2}{2k_B T}\right), \quad (29)$$

which corresponds to the stationary distribution.

C. The Fokker-Planck-Kramers equation for free Brownian motion

The stochastic differential equation for free Brownian motion taking into account the position $\mathbf{r}=(x,y,z)$ and the velocity $\mathbf{u}=(u_x,u_y,u_z)$ is

$$\dot{\mathbf{r}} = \mathbf{u},$$

$$\dot{\mathbf{u}} = -\beta \mathbf{u} + \mathbf{A}(t), \quad (30)$$

where $\mathbf{A}(t)$ satisfies the same properties given in Sec. II A. This set of six stochastic differential equations can be written as the Ornstein-Uhlenbeck process (10), in such a way that the drift matrix Υ and the diffusion matrix $\mathbb{D}$ are given by

$$\Upsilon = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \end{pmatrix}, \quad (31)$$

$$\mathbb{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}.$$

In this case we have the vector $\{x\}=(x_1,x_2,x_3,x_4,x_5,x_6)=(x,y,z,u_x,u_y,u_z)$. From Eq. (3) we construct the Fokker-Planck-Kramers equation in the absence of a external force for the transition probability $P(\mathbf{r},\mathbf{u},t|\mathbf{u}_0,\mathbf{r}_0)$ governing the probability of the simultaneous occurrence of the velocity $\mathbf{u}$ and the position $\mathbf{r}$ at time t given that $\mathbf{u}=\mathbf{u}_0$ and $\mathbf{r}=\mathbf{r}_0$ at $t=0$, which can be written as

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x_i} u_i P + \beta \frac{\partial}{\partial u_i} u_i P + q \frac{\partial^2}{\partial u_i^2} P, \quad (32)$$

or

$$\frac{\partial P}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{r}} P = \beta \,\mathrm{div}_{\mathbf{u}}(\mathbf{u}P) + q\nabla_{\mathbf{u}}^2 P. \quad (33)$$

The initial condition (6) in this case will be

$$\lim_{t\to 0} P(\mathbf{r},\mathbf{u},t|\mathbf{u}_0,\mathbf{r}_0) \equiv \delta^3(\mathbf{u}-\mathbf{u}_0)\,\delta^3(\mathbf{r}-\mathbf{r}_0). \quad (34)$$

Again, following Chandrasekhar's proposal, the solution of Eq. (33) is connected with the solution of the first-order equation

$$\frac{\partial P}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{r}} P - \beta \,\mathrm{div}_{\mathbf{u}}(\mathbf{u}P) = 0. \quad (35)$$

The general solution of this equation can expressed in terms of six independents integrals of the Lagrangian subsidiary system

$$\dot{\mathbf{u}} = -\beta \mathbf{u}, \quad \dot{\mathbf{r}} = \mathbf{u}, \quad (36)$$

namely

$$\mathbf{u} e^{\beta t} = \mathbf{I}_1, \quad \mathbf{r} + \beta^{-1}\mathbf{u} = \mathbf{I}_2, \quad (37)$$

where the constants $\mathbf{I}_1$ and $\mathbf{I}_2$ are defined as $\mathbf{I}_1=\mathbf{u}_0$ and $\mathbf{I}_2=\mathbf{r}_0+\beta^{-1}\mathbf{u}_0$. If we define $P(\mathbf{R},\mathbf{S}) \equiv P(\mathbf{r},\mathbf{u},t|\mathbf{u}_0,\mathbf{r}_0)$, where

$$\mathbf{R} = \mathbf{r} - \mathbf{r}_0 - \frac{\mathbf{u}_0}{\beta}(1-e^{-\beta t}),$$

$$\mathbf{S} = \mathbf{u} - \mathbf{u}_0 e^{-\beta t}, \quad (38)$$

then the general solution of Eq. (33) can be expressed in the form

$$P(\mathbf{R},\mathbf{S}) = \frac{1}{8\pi^3(FG-H^2)^{3/2}} \times \exp\left(\frac{-(F|\mathbf{S}|^2 - 2H\mathbf{R}\cdot\mathbf{S} + G|\mathbf{R}|^2)}{2(FG-H^2)}\right), \quad (39)$$

where

$$F = \frac{q}{\beta^3}(2\beta t - 3 + 4e^{-\beta t} - e^{-2\beta t}),$$

$$G = \frac{q}{\beta}(1 - e^{-2\beta t}),$$

$$H = \frac{q}{\beta^2}(1 - e^{-\beta t})^2,$$

$$FG - H^2 = (ab - h^2)e^{-2\beta t}, \quad (40)$$

together with the parameters

$$a = 2\frac{q}{\beta^2}t, \quad b = \frac{q}{\beta}(e^{2\beta t} - 1), \quad h = \frac{2q}{\beta^2}(1 - e^{\beta t}). \quad (41)$$

The transition probability $P(\mathbf{R}) \equiv P(\mathbf{r},t|\mathbf{r}_0,\mathbf{u}_0)$ governing the probability of the position $\mathbf{r}$ at time t, given that the particle is at $\mathbf{r}_0$ with a velocity $\mathbf{u}_0$ at $t=0$, can be calculated through the integral

$$P(\mathbf{R}) \equiv P(\mathbf{r},t|\mathbf{r}_0,\mathbf{u}_0) = \int P(\mathbf{R},\mathbf{S})d\mathbf{S}. \quad (42)$$

After some algebra the result is

$$P(\mathbf{r},t|\mathbf{r}_0,\mathbf{u}_0) = \left(\frac{\beta^3}{2\pi q(2\beta t - 3 + 4e^{-\beta t} - e^{-2\beta t})}\right)^{3/2} \times \exp\left(-\frac{\beta^3|\mathbf{r} - \mathbf{r}_0 - \beta^{-1}\mathbf{u}_0(1 - e^{-\beta t})|^2}{2q(2\beta t - 3 + 4e^{-\beta t} - e^{-2\beta t})}\right). \quad (43)$$

For large times such that $\beta t \gg 1$, it reduces to

$$P(\mathbf{r},t|\mathbf{r}_0,\mathbf{u}_0) \simeq \frac{1}{(4\pi Dt)^{3/2}} \exp\left(-\frac{|\mathbf{r} - \mathbf{r}_0|^2}{4Dt}\right), \quad (44)$$

D being the Einstein' diffusion constant given by $D = q/\beta^2 = k_B T/m\beta$.

III. THE FOKKER-PLANCK EQUATION FOR A CHARGED PARTICLE IN A MAGNETIC FIELD

The Langevin equation (20) can be adapted to the case of diffusion of charged particles (Brownian gas) in the presence of a constant magnetic field that acts upon particles via the Lorentz force. In this case the corresponding Langevin equation is [2,4]

$$\dot{\mathbf{u}} = -\beta\mathbf{u} + \frac{e}{mc}\mathbf{u} \times \mathbf{B} + \mathbf{A}(t), \quad (45)$$

where e denotes the charge of the particle of mass m and $\mathbf{A}(t)$ satisfies the same properties as that given in Eq. (21). We assume for simplicity that the constant magnetic field $\mathbf{B}$ is directed along the z axis of a Cartesian reference frame, that is $\mathbf{B} = (0,0,B)$ with B a constant. In this case the above equation can also be written as

$$\dot{\mathbf{u}} = -\beta\mathbf{u} + \mathbb{W}\mathbf{u} + \mathbf{A}(t), \quad (46)$$

where $\mathbb{W}$ is a real antisymmetric matrix given by

$$\mathbb{W} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (47)$$

$\omega = eB/mc$ being the Larmor frequency. To establish the Fokker-Planck equation associated to Eq. (46), this last one must be written as

$$\dot{\mathbf{u}} = -\Lambda\mathbf{u} + \mathbf{A}(t), \quad (48)$$

where Λ now reads as

$$\Lambda = \begin{pmatrix} \beta & -\omega & 0 \\ \omega & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}. \quad (49)$$

Equation (48) is a coupled system of equations in the plane (u_x, u_y) and independent from the coordinate u_z, for which the corresponding evolution equation is the Langevin equation of the ordinary Brownian motion. The drift and diffusion coefficients are

$$D_i = -\Lambda_{ij}u_j,$$

$$D_{ij} = q\delta_{ij}, \quad (50)$$

and therefore the associated Fokker-Planck equation is

$$\frac{\partial P}{\partial t} = \mathrm{div}_{\mathbf{u}}(\Lambda\mathbf{u}P) + q\nabla^2_{\mathbf{u}}P. \quad (51)$$

To solve this equation by following Chandrasekhar's idea we require to solve the first-order equation of Eq. (51) without the Laplacian term, which involves the Lagrangian subsidiary system

$$\dot{\mathbf{u}} = -\Lambda\mathbf{u}, \quad (52)$$

together with the required first integrals

$$\mathbf{u}e^{\Lambda t} = \mathbf{u}_0 = \mathbf{I}_1 = \text{const}. \quad (53)$$

However, such a solution is not easy to calculate because the resulting equations are also a coupled system. To solve this problem we propose the following strategy. If we make the change of variables $\mathbf{u}' = e^{-\mathbb{W}t}\mathbf{u}$, then the Langevin equation (46), in the new velocity space, takes the very simple form

$$\dot{\mathbf{u}}' = -\beta\mathbf{u}' + \mathbb{R}^{-1}(t)\mathbf{A}(t), \quad (54)$$

where $\mathbb{R}(t) = e^{\mathbb{W}t}$ is an orthogonal rotation matrix given by

$$\mathbb{R}(t) = \begin{pmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{55}$$

in such a way that $\mathbb{R}^T(t) = \mathbb{R}^{-1}(t)$, i.e., the transposed is its inverse and therefore $\mathbb{R}^{-1}(t) = e^{-\mathbb{W}t}$. As we can see, the Langevin equation (54) is the same as that of the free Brownian motion except by the factor $\mathbb{R}^{-1}(t)$ multiplying the noise $\mathbf{A}(t)$, which amounts to a rotation of the noise $\mathbf{A}(t)$. We can observe immediately from Eq. (54) that the drift and the diffusion coefficients are

$$D'_i = -\beta u'_i,$$

$$D'_{ij} = q(\mathbb{R}^{-1}(t))_{ik}(\mathbb{R}^{-1}(t))_{jk} = q\delta_{ij}. \tag{56}$$

As we can see, the diffusion coefficient D'_{ij} gives the same result as that given in Eq. (50) because $(\mathbb{R}^{-1}(t))_{ik}(\mathbb{R}^{-1}(t))_{jk} = \delta_{ij}$. The interesting point that we can remark from this result is the following: the term $\mathbb{R}^{-1}(t)\mathbf{A}(t)$, which represents a rotation of the noise $\mathbf{A}(t)$, has the same statistical properties as $\mathbf{A}(t)$ if this last one satisfies the property (21), being the reason why D'_{ij} is the same as D_{ij}. This fact physically means that the rotation matrix with elements $\sin \omega t$ and $\cos \omega t$ has been absorbed by the noise $\mathbf{A}(t)$ and therefore such oscillating functions will not appear any more in the diffusion constant of the associated Fokker-Planck equation. On the other hand, the drift and diffusion coefficients given in Eq. (50) can also be derived from the transformations (16) and (17).

The Fokker-Planck equation for the transition probability density of the velocity conditioned by initial data $\mathbf{u}'_0$ at time $t_0=0$, i.e., $P'(\mathbf{u}',t|\mathbf{u}'_0)$, is then

$$\frac{\partial P'}{\partial t} = \beta\, \mathrm{div}_{\mathbf{u}'}(\mathbf{u}'P') + q\nabla^2_{\mathbf{u}'}P'. \tag{57}$$

Because $P'=JP$, in this case it can be shown that the Jacobian is $J=1$. Then the initial condition for P' also satisfies

$$\lim_{t\to 0} P'(\mathbf{u}',t|\mathbf{u}'_0) \equiv \delta^3(\mathbf{u}'-\mathbf{u}'_0) = \delta(u'_1 - u'_{01})\,\delta(u'_2 - u'_{02})\,\delta(u'_3 - u'_{03}), \tag{58}$$

the same as in the ordinary Brownian motion. The solution of Eq. (57) is connected with the solution of the associated first-order equation, which involves the first three integrals of the Lagrangian subsidiary system

$$\dot{\mathbf{u}}' = -\beta \mathbf{u}', \tag{59}$$

which are

$$\mathbf{u}' e^{\beta t} = \mathbf{u}'_0 = \mathbf{I}'_1 = \text{const}. \tag{60}$$

Finally, the solution of Eq. (57) reads as

$$P'(\mathbf{u}',t|\mathbf{u}'_0) = \frac{1}{[2\pi q(1-e^{-2\beta t})/\beta]^{3/2}} \times \exp\left(-\frac{\beta|\mathbf{u}'-\mathbf{u}'_0 e^{-\beta t}|^2}{2q(1-e^{-2\beta t})}\right). \tag{61}$$

Returning to the variables $\mathbf{u}$, we first notice that

$$|\mathbf{u}'-\mathbf{u}'_0 e^{-\beta t}|^2 \equiv [\mathbf{u}'-\mathbf{u}'_0 e^{-\beta t}]\cdot[\mathbf{u}'-\mathbf{u}'_0 e^{-\beta t}] = |\mathbf{u} - e^{-\Lambda t}\mathbf{u}_0|^2, \tag{62}$$

where Λ is the same as in Eq. (49). Therefore, in the $\mathbf{u}$ velocity space the transition probability density is given by

$$P(\mathbf{u},t|\mathbf{u}_0) = \frac{1}{[2\pi q(1-e^{-2\beta t})/\beta]^{3/2}} \times \exp\left(-\frac{\beta|\mathbf{u}-e^{-\Lambda t}\mathbf{u}_0|^2}{2q(1-e^{-2\beta t})}\right), \tag{63}$$

which is exactly the same as that calculated by Czopnik and Garbaczewski [4] by other method. Also, in the long-time limit, it reduces to the Maxwellian distribution

$$P(\mathbf{u}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m|\mathbf{u}|^2}{2k_B T}\right). \tag{64}$$

Since the process is Markovian, then Eq. (63) can be easily extended to the case of arbitrary $t_0 = t' \neq 0$. For this case $P(\mathbf{u},t|\mathbf{u}',t')$ can be derived by substituting anywhere $t-t'$ instead of t and $\mathbf{u}'$ instead of $\mathbf{u}_0$, where $\mathbf{u}'=\mathbf{u}(t')$ (*here we must not confuse this* $\mathbf{u}'$ *with the transformation of variables defined earlier*). In the stationary state the joint probability density $W_2(\mathbf{u},t;\mathbf{u}',t')$ may be expressed by the product of the aforementioned transition probability and Eq. (64). So, for both $t \geq t'$ and $t \leq t'$, it can be written as

$$W_2(\mathbf{u},t;\mathbf{u}',t') = \left(\frac{\beta}{2\pi q}\right)^3 \frac{1}{[(1-e^{-2\beta|t-t'|})]^{3/2}} \times \exp\left(-\frac{\beta[|\mathbf{u}|^2 - 2\mathbf{u}\cdot\mathbf{u}' e^{-\Lambda|t-t'|} + |\mathbf{u}'|^2]}{2q(1-e^{-2\beta|t-t'|})}\right). \tag{65}$$

This joint probability density uniquely determines a stationary Markovian stochastic process for which we can calculate various mean values. For instance, the mean values of the velocity components $u_i(t)$ for $i=1,2,3$ are equal to zero, that is,

$$\langle u_i(t)\rangle = \int_{-\infty}^{\infty} u_i P(\mathbf{u})\, d\mathbf{u} = 0. \tag{66}$$

The matrix of the second moments (velocity autocorrelation functions) reads

$$\langle u_i(t) u_j(t')\rangle = \int_{-\infty}^{\infty} u_i u'_j W_2(\mathbf{u},t;\mathbf{u}',t')\, d\mathbf{u}\, d\mathbf{u}', \tag{67}$$

with $i,j=1,2,3$. For instance, the autocorrelation function of the x component of the velocity yields

$$\langle u_1(t)u_1(t')\rangle = \frac{q}{\beta}e^{-\beta|t-t'|}\cos\omega|t-t'|, \tag{68}$$

with the same expression for the y component. On the other hand, the z component satisfies the usual expression for free Brownian motion, i.e., $\langle u_3(t)u_3(t')\rangle=(q/\beta)e^{-\beta|t-t'|}$.

From these autocorrelation functions, the mean square displacement (MSD) for the position components $[\mathbf{r}\equiv(r_1,r_2,r_3)\equiv(x,y,z)]$ can easily be calculated if the particle starts at time $t=0$ at $\mathbf{r}=\mathbf{r}_0$ with the velocity $\mathbf{u}=\mathbf{u}_0$. The x component is

$$\langle(\Delta x')^2\rangle = 2D\left(\frac{\beta^2}{\omega^2+\beta^2}\right)t - \frac{q}{\beta\Lambda_1^2}(1-e^{-\Lambda_1 t}) - \frac{q}{\beta\Lambda_2^2}(1-e^{-\Lambda_2 t}), \tag{69}$$

where $D=q/\beta^2=k_BT/m\beta$, which is consistent with the value of the Einstein's diffusion constant, $\Lambda_1=\beta-i\omega$, and $\Lambda_2=\beta+i\omega$. The same result is obtained for the y component. The z component reads

$$\langle(\Delta z)^2\rangle = 2Dt - \frac{2q}{\beta^3}(1-e^{-\beta t}). \tag{70}$$

An analysis of large times ($\beta t \gg 1$) shows that the MSD across the magnetic field is

$$\langle(\Delta x)^2\rangle = \langle(\Delta y)^2\rangle = 2D\left(\frac{\beta^2}{\omega^2+\beta^2}\right)t, \tag{71}$$

and along the magnetic field

$$\langle(\Delta z)^2\rangle = 2Dt, \tag{72}$$

where $D=q/\beta^2=k_BT/m\beta$ is the Einstein's diffusion constant. Equations (71) and (72) agree with those calculated by Kurşunoğlu.

IV. THE FOKKER-PLANCK-KRAMERS EQUATION FOR A CHARGED PARTICLE IN A MAGNETIC FIELD

A. General TPD through the Fokker-Planck formalism

In phase space the diffusion of charged particles in the absence of external force can be described by the following set of stochastic differential equations

$$\dot{\mathbf{r}} = \mathbf{u},$$

$$\dot{\mathbf{u}} = -\beta\mathbf{u} + \mathbb{W}\mathbf{u} + \mathbf{A}(t), \tag{73}$$

or

$$\dot{\mathbf{r}} = \mathbf{u},$$

$$\dot{\mathbf{u}} = -\Lambda\mathbf{u} + \mathbf{A}(t). \tag{74}$$

The Fokker-Planck-Kramers equation associated with the system (74) is then

$$\frac{\partial P}{\partial t} + \mathbf{u}\cdot\nabla_{\mathbf{r}}P = \mathrm{div}_{\mathbf{u}}(\Lambda\mathbf{u}P) + q\nabla_{\mathbf{u}}^2 P. \tag{75}$$

Proceeding as in the previous section, the first-order solution may be expressed in terms of six integrals of the Lagrangian subsidiary system

$$\dot{\mathbf{u}} = -\Lambda\mathbf{u}, \quad \dot{\mathbf{r}} = \mathbf{u}. \tag{76}$$

These integrals are

$$\mathbf{u}e^{\Lambda t} = \mathbf{I}_1, \quad \mathbf{r} + \Lambda^{-1}\mathbf{u} = \mathbf{I}_2, \tag{77}$$

where now $\mathbf{I}_1=\mathbf{u}_0$, $\mathbf{I}_2=\mathbf{r}_0+\Lambda^{-1}\mathbf{u}_0$, and Λ^{-1} is the inverse of matrix Λ given by

$$\Lambda^{-1} = \begin{pmatrix} \frac{\beta}{\omega^2+\beta^2} & \frac{\omega}{\omega^2+\beta^2} & 0 \\ -\frac{\omega}{\omega^2+\beta^2} & \frac{\beta}{\omega^2+\beta^2} & 0 \\ 0 & 0 & \frac{1}{\beta} \end{pmatrix}. \tag{78}$$

However, as before, the integrals (77) lead to a set of coupled equations which are not immediate to solve. To proceed further, we propose to transform the Langevin equations (73), given in the phase space $(\mathbf{r},\mathbf{u})$ to another phase space $(\mathbf{r}',\mathbf{u}')$ in which the resulting Langevin equations are very similar to those of ordinary Brownian motion. This can be achieved if we define $\dot{\mathbf{r}}'=\mathbf{u}'$ in such a way that

$$\mathbf{u}' = e^{-\mathbb{W}t}\mathbf{u}, \quad \dot{\mathbf{r}}' = e^{-\mathbb{W}t}\dot{\mathbf{r}}. \tag{79}$$

In this case Eq. (73) is transformed into

$$\dot{\mathbf{r}}' = \mathbf{u}',$$

$$\dot{\mathbf{u}}' = -\beta\mathbf{u} + \mathbb{R}(t)^{-1}\mathbf{A}(t), \tag{80}$$

which correspond to a set of decoupled stochastic differential equations in the transformed phase space and are very similar to those of free Brownian motion. Then, as in Sec. II, the drift matrix Υ' and the diffusion matrix $\mathbb{D}'$ associated with Eq. (80) are exactly the same as that given in Eq. (31). So, the Fokker-Planck-Kramers equation is

$$\frac{\partial P'}{\partial t} + \mathbf{u}'\cdot\nabla_{\mathbf{r}'}P' = \beta\,\mathrm{div}_{\mathbf{u}'}(\mathbf{u}'P') + q\nabla_{\mathbf{u}'}^2 P'. \tag{81}$$

Again, as in the ordinary Brownian motion, the Lagrangian subsidiary system associated with the first-order equation without the Laplacian term of Eq. (81) is

$$\dot{\mathbf{u}}' = -\beta\mathbf{u}', \quad \dot{\mathbf{r}}' = \mathbf{u}', \tag{82}$$

and their corresponding six integrals are

$$\mathbf{u}'e^{\beta t} = \mathbf{I}_1', \quad \mathbf{r}' + \beta^{-1}\mathbf{u}' = \mathbf{I}_2', \tag{83}$$

where $\mathbf{I}_1'=\mathbf{u}_0'$ and $\mathbf{I}_2'=\mathbf{r}_0'+\beta^{-1}\mathbf{u}_0'$.

Before proceeding, we must establish the transformation which makes the connection between the representations $(\mathbf{r},\mathbf{u})$ and $(\mathbf{r}',\mathbf{u}')$ and which is also consistent with the transformation (80). The desired transformation can be obtained with the aid of Eqs. (77) and (83), together with the transformation $\mathbf{u}'=e^{-\mathbb{W}t}\mathbf{u}$, such that

$$\mathbf{r}' = \beta^{-1}e^{-\mathbb{W}t}\Lambda(\mathbf{r}-\mathbf{I}_2)+\mathbf{I}_2'. \tag{84}$$

If we use this result together with the transformation for $\mathbf{u}'$ in Eqs. (16) and (17) we obtain the same drift Υ' and diffusion $\mathbb{D}'$ matrices as those given in Eq. (31). Under these circumstances, if we define $P'(\mathbf{R}',\mathbf{S}')\equiv P'(\mathbf{r}',\mathbf{u}',t|\mathbf{u}_0',\mathbf{r}_0')$, then the solution of Eq. (81), with the initial condition

$$\lim_{t\to 0} P'(\mathbf{r}',\mathbf{u}',t|\mathbf{u}_0',\mathbf{r}_0') \equiv \delta^3(\mathbf{u}'-\mathbf{u}_0')\delta^3(\mathbf{r}'-\mathbf{r}_0'), \tag{85}$$

can be written in the same form as Eq. (39), i.e.,

$$P'(\mathbf{R}',\mathbf{S}') = \frac{1}{8\pi^3(FG-H^2)^{3/2}} \times \exp\left(\frac{-(F|\mathbf{S}'|^2-2H\mathbf{R}'\cdot\mathbf{S}'+G|\mathbf{R}'|^2)}{2(FG-H^2)}\right), \tag{86}$$

where now

$$\mathbf{R}' = \mathbf{r}'-\mathbf{r}_0'-\frac{\mathbf{u}_0'}{\beta}(1-e^{-\beta t}),$$

$$\mathbf{S}' = \mathbf{u}'-\mathbf{u}_0'e^{-\beta t}, \tag{87}$$

and the parameters F, G, and H are the same as in Eqs. (40) and (41).

Returning to the transition probability density P we use the fact that $Pd\mathbf{S}d\mathbf{R}=P'd\mathbf{S}'d\mathbf{R}'$, where the volume element transforms as

$$d\mathbf{S}d\mathbf{R} = Jd\mathbf{S}'d\mathbf{R}', \tag{88}$$

J being the Jacobian of the transformation and therefore $JP=P'$. It can be shown that $J=J_SJ_R$, where

$$J_S \equiv |\mathrm{Det}(\partial S_i/\partial S_j')| = 1/J_S' = 1/|\mathrm{Det}(\partial S_i'/\partial S_j)|. \tag{89}$$

Similarly

$$J_R \equiv |\mathrm{Det}(\partial R_i/\partial R_j')| = 1/J_R' = 1/|\mathrm{Det}(\partial R_i'/\partial R_j)|. \tag{90}$$

Then, $J=J_SJ_R=1/J_S'J_R'=1/J'$. The explicit form of these transformations can be calculated if we first observe from Eq. (87) that

$$\mathbf{S}' = \mathbf{u}'-\mathbf{u}_0'e^{-\beta t} = e^{-\mathbb{W}t}\mathbf{S}, \tag{91}$$

where we define

$$\mathbf{S} = \mathbf{u}-e^{-\Lambda t}\mathbf{u}_0. \tag{92}$$

For $\mathbf{R}'$ we use Eq. (84) to show that

$$\mathbf{R}' = \mathbf{r}'-\mathbf{r}_0'-\frac{\mathbf{u}_0'}{\beta}(1-e^{-\beta t}) = e^{-\mathbb{W}t}\Lambda\mathbf{R}, \tag{93}$$

with

$$\mathbf{R} = \mathbf{r}-\mathbf{r}_0-\Omega\mathbf{u}_0, \tag{94}$$

and $\Omega\equiv\Lambda^{-1}(1-e^{-\Lambda t})$. It can be shown, from Eqs. (91) and (93) that

$$J_S' = 1, \quad J_R' = \frac{\omega^2+\beta^2}{\beta^2}, \tag{95}$$

and therefore $P=J_R'P'$. On the other hand, if we define the vectors on the xy plane as $\tilde{\mathbf{S}}=(S_1,S_2)$ and $\tilde{\mathbf{R}}=(R_1,R_2)$, then $\mathbf{S}=(\tilde{\mathbf{S}},S_3)$ and $\mathbf{R}=(\tilde{\mathbf{R}},R_3)$, where S_3 and R_3 are the z components of the vectors $\mathbf{S}$ and $\mathbf{R}$, such that $S_3=u_z-u_{0z}e^{-\beta t}$ and $R_3=z-z_0-\beta^{-1}u_{0z}(1-e^{-\beta t})$, with $u_{0z}=u_z(0)$ and $z_0=z(0)$. From these definitions, the vectors $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{R}}$ can be written as

$$\tilde{\mathbf{S}} = \tilde{\mathbf{u}}-e^{-\tilde{\Lambda}t}\tilde{\mathbf{u}}_0, \quad \tilde{\mathbf{R}} = \tilde{\mathbf{r}}-\tilde{\mathbf{r}}_0-\tilde{\Omega}\tilde{\mathbf{u}}_0, \tag{96}$$

where $\tilde{\Omega}\equiv\tilde{\Lambda}^{-1}(1-e^{-\tilde{\Lambda}t})$, with the matrices

$$\tilde{\Lambda} = \begin{pmatrix} \beta & -\omega \\ \omega & \beta \end{pmatrix}, \quad \tilde{\Lambda}^{-1} = \begin{pmatrix} \dfrac{\beta}{\omega^2+\beta^2} & \dfrac{\omega}{\omega^2+\beta^2} \\ -\dfrac{\omega}{\omega^2+\beta^2} & \dfrac{\beta}{\omega^2+\beta^2} \end{pmatrix}. \tag{97}$$

Under these circumstances, it can be shown that

$$|\mathbf{S}'|^2 = |\mathbf{S}|^2 = |\tilde{\mathbf{S}}|^2 + S_3^2, \tag{98}$$

$$|\mathbf{R}'|^2 = \delta|\tilde{\mathbf{R}}|^2 + R_3^2, \tag{99}$$

$$\mathbf{S}'\cdot\mathbf{R}' = \tilde{\mathbf{S}}\cdot\tilde{\mathbf{R}} + \frac{\omega}{\beta}(\tilde{\mathbf{S}}\times\tilde{\mathbf{R}})_z + S_3R_3, \tag{100}$$

where $(\tilde{\mathbf{S}}\times\tilde{\mathbf{R}})_z=(S_1R_2-S_2R_1)$ is the z component of the cross product and $\delta=(\omega^2+\beta^2)/\beta^2$. Substituting Eqs. (98)–(100) into Eq. (86) we can verify that the TPD for P can be written as the product of two independent transition probability densities, that is $P(\mathbf{R},\mathbf{S})\equiv\tilde{P}(\tilde{\mathbf{R}},\tilde{\mathbf{S}})P_z(z,u_z,t|u_{0z},z_0)$, where

$$\tilde{P}(\tilde{\mathbf{R}},\tilde{\mathbf{S}}) = \frac{J_R'}{4\pi^2(FG-H^2)} \times\exp\left(\frac{-\left(F|\tilde{\mathbf{S}}|^2-2H\tilde{\mathbf{R}}\cdot\tilde{\mathbf{S}}+2\left(\frac{\omega}{\beta}\right)H(\tilde{\mathbf{S}}\times\tilde{\mathbf{R}})_z+\delta G|\tilde{\mathbf{R}}|^2\right)}{2(FG-H^2)}\right) \tag{101}$$

corresponds to the planar TPD which describes the diffusion process on the xy plane and

$$P_z(z,u_z,t|u_{0z},z_0) = \frac{1}{[4\pi^2(FG-H^2)]^{1/2}} \times\exp\left(-\frac{FS_3^2-2HR_3S_3+GR_3^2}{2(FG-H^2)}\right), \tag{102}$$

is the TPD which describes the diffusion process along the z axis and is the same as that of the ordinary Brownian motion, as expected. From Eq. (101), we can calculate the spatial transition probability density $\tilde{P}(\tilde{\mathbf{R}})\equiv\tilde{P}(\tilde{\mathbf{r}},t|\tilde{\mathbf{r}}_0,\tilde{\mathbf{u}}_0)$ through the integral

$$\tilde{P}(\tilde{\mathbf{R}}) = \int \tilde{P}(\tilde{\mathbf{S}},\tilde{\mathbf{R}})d\tilde{\mathbf{S}}. \tag{103}$$

After a long but straightforward algebra it reduces to

$$\tilde{P}(\tilde{\mathbf{r}},t|\tilde{\mathbf{r}}_0,\tilde{\mathbf{u}}_0) = \frac{1}{2\pi(D_e/\beta)(2\beta t - 3 + 4e^{-\beta t} - e^{-2\beta t})} \times \exp\left(-\frac{|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}_0 - \tilde{\Omega}\tilde{\mathbf{u}}_0|^2}{2(D_e/\beta)(2\beta t - 3 + 4e^{-\beta t} - e^{-2\beta t})}\right), \tag{104}$$

where $D_e = D\beta^2/(\omega^2+\beta^2)$, which corresponds to a normalization of the diffusion constant D. For large times such that $\beta t \gg 1$, it immediately reduces to

$$\tilde{P}(\tilde{\mathbf{r}},t|\tilde{\mathbf{r}}_0,\tilde{\mathbf{u}}_0) \simeq \frac{1}{(4\pi D_e t)} \exp\left(-\frac{|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}_0|^2}{4D_e t}\right). \tag{105}$$

The MSD for the x and y components can also be calculated from Eq. (105), with a result identical to the expression given by Eq. (71), that is,

$$\langle(\Delta x)^2\rangle = \langle(\Delta y)^2\rangle = 2D\left(\frac{\beta^2}{\omega^2+\beta^2}\right)t. \tag{106}$$

Evidently, $\tilde{P}(\tilde{\mathbf{S}}) \equiv P(\tilde{\mathbf{u}},t|\tilde{\mathbf{u}}_0)$ can also be obtained from the integral

$$\tilde{P}(\tilde{\mathbf{S}}) = \int \tilde{P}(\tilde{\mathbf{S}},\tilde{\mathbf{R}})d\tilde{\mathbf{R}}, \tag{107}$$

with the result

$$\tilde{P}(\tilde{\mathbf{u}},t|\tilde{\mathbf{u}}_0) = \frac{1}{[2\pi q(1-e^{-2\beta t})/\beta]} \times \exp\left(-\frac{|\tilde{\mathbf{u}} - e^{-\tilde{\Lambda}t}\tilde{\mathbf{u}}_0|^2}{2q(1-e^{-2\beta t})/\beta}\right), \tag{108}$$

which corresponds to the planar TPD for the velocity space. Clearly, Eq. (63) can also be written as the product of two independent TPD; one is the TPD (108) and the other is

$$P_z(u_z,t|u_{0z}) = \frac{1}{[2\pi q(1-e^{-2\beta t})/\beta]^{1/2}} \times \exp\left(-\frac{(u_z - u_{0z}e^{-\beta t})^2}{2q(1-e^{-2\beta t})/\beta}\right), \tag{109}$$

corresponding to the ordinary Brownian motion.

In the following section we will show that the general TPD (101) is equivalent to the joint probability distribution (JPD) $\tilde{P}(\tilde{\mathbf{R}},\tilde{\mathbf{S}})$ of the vectors $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{S}}$ defined above.

B. General planar JPD through the correlation matrix formalism

The interesting point of the general expression (101) is that, although very similar, it is not the same to the calculated by Czopnik and Garbaczewski. To show this argument, we will use the same strategy used by these authors to calculate the planar JPD $\tilde{P}'(\tilde{\mathbf{R}}',\tilde{\mathbf{S}}')$, where now the vectors $\tilde{\mathbf{R}}'$ and $\tilde{\mathbf{S}}'$ are defined on the $x'y'$ plane, that is $\tilde{\mathbf{R}}' = (R_1', R_2')$ and $\tilde{\mathbf{S}}' = (S_1', S_2')$, and have the same expressions as that given by Eq. (87), i.e.,

$$\tilde{\mathbf{R}}' = \tilde{\mathbf{r}}' - \tilde{\mathbf{r}}_0' - \frac{\tilde{\mathbf{u}}_0'}{\beta}(1-e^{-\beta t}),$$

$$\tilde{\mathbf{S}}' = \tilde{\mathbf{u}}' - \tilde{\mathbf{u}}_0' e^{-\beta t}. \tag{110}$$

We can easily see from Eq. (80) that

$$\tilde{\mathbf{S}}' = e^{-\beta t}\int_0^t e^{\beta s}\tilde{\mathbf{A}}'(s)ds, \tag{111}$$

$$\tilde{\mathbf{R}}' = -\beta^{-1}e^{-\beta t}\int_0^t e^{\beta s}\tilde{\mathbf{A}}'(s)ds + \beta^{-1}\int_0^t \tilde{\mathbf{A}}'(s)ds, \tag{112}$$

where $\tilde{\mathbf{A}}'(t) = \tilde{\mathbb{R}}^{-1}(t)\tilde{\mathbf{A}}(t)$, which corresponds to the rotation of the noise $\tilde{\mathbf{A}}(t)$, and $\tilde{\mathbb{R}}^{-1}(t)$ is a 2×2 rotation matrix without the third row and the third column of matrix given in Eq. (55). Both vectors $\tilde{\mathbf{S}}'$ and $\tilde{\mathbf{R}}'$ correspond to Gaussian distributions, each one with zero mean value. Therefore the planar JPD is given by

$$\tilde{P}'(\tilde{\mathbf{R}}',\tilde{\mathbf{S}}') = \frac{1}{4\pi^2(\det C)^{1/2}} \exp\left(-\frac{1}{2}\sum_{i,j} c_{ij}^{-1}x_i x_j\right), \tag{113}$$

where $C = c_{ij} = \langle x_i x_j\rangle$ is the matrix of variances and covariances, such that the variable $\mathbf{x} = (x_1, x_2, x_3, x_4) \equiv (R_1', R_2', S_1', S_2')$ presents a vector in the four-dimensional phase space and c_{ij}^{-1} represents the components of the inverse matrix C^{-1}. From Eq. (111) it can be shown that

$$G = \langle S_1'S_1'\rangle = \langle S_2'S_2'\rangle = \frac{q}{\beta}(1-e^{-2\beta t}), \tag{114}$$

while $\langle S_1'S_2'\rangle = \langle S_2'S_1'\rangle = 0$. From Eq. (112) we have

$$F = \langle R_1'R_1'\rangle = \langle R_2'R_2'\rangle = \frac{q}{\beta^3}(2\beta t - 3 + 4e^{-\beta t} - e^{-2\beta t}) \tag{115}$$

and $\langle R_1'R_2'\rangle = \langle R_2'R_1'\rangle = 0$. We also show that

$$H = \langle R_1'S_1'\rangle = \langle R_2'S_2'\rangle = \frac{q}{\beta^2}(1-e^{-\beta t})^2 \tag{116}$$

and $\langle R_1'S_2'\rangle = \langle R_2'S_1'\rangle = 0$. Thus, the covariance matrix $C = c_{ij}$ reads as

$$C = \begin{pmatrix} G & 0 & H & 0 \\ 0 & G & 0 & H \\ H & 0 & F & 0 \\ 0 & H & 0 & F \end{pmatrix}, \tag{117}$$

and its inverse matrix will be

$$C^{-1} = \frac{FG-H^2}{\det C}\begin{pmatrix} F & 0 & -H & 0 \\ 0 & F & 0 & -H \\ -H & 0 & G & 0 \\ 0 & -H & 0 & G \end{pmatrix}, \quad (118)$$

where $\det C=(FG-H^2)^2$. Therefore, the planar JPD of Eq. (113) can be written as

$$\tilde{P}'(\tilde{\mathbf{R}}',\tilde{\mathbf{S}}') = \frac{1}{4\pi^2(FG-H^2)} \times\exp\left(\frac{-(F|\tilde{\mathbf{S}}'|^2 - 2H\tilde{\mathbf{R}}'\cdot\tilde{\mathbf{S}}' + G\tilde{\mathbf{R}}'|)}{(FG-H^2)}\right), \quad (119)$$

which is exactly the same as that of ordinary Brownian motion with parameters F, G, and H the same as those given by Eq. (40), as expected. To return to the planar JPD $\tilde{P}(\tilde{\mathbf{R}},\tilde{\mathbf{S}})$ we follow the same algebra given before. So, according to Eqs. (91) and (93), the vectors $\tilde{\mathbf{S}}'$ and $\tilde{\mathbf{R}}'$ can also be written as

$$\tilde{\mathbf{S}}' = e^{-\tilde{\mathbb{W}}t}\tilde{\mathbf{S}}, \quad (120)$$

$$\tilde{\mathbf{R}}' = e^{-\tilde{\mathbb{W}}t}\tilde{\Lambda}\tilde{\mathbf{R}}, \quad (121)$$

where $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{R}}$ are the same as those given in Eq. (96) and $\tilde{\mathbb{W}}$ is the 2×2 antisymmetric matrix, i.e., without the third row and the third column of matrix given in Eq. (47). In this case it can easily shown that $|\tilde{\mathbf{S}}'|^2$, $|\tilde{\mathbf{R}}'|^2$, and $\tilde{\mathbf{R}}'\cdot\tilde{\mathbf{S}}'$ are, respectively, the same as those given by Eqs. (98)–(100) without the components S_3 and R_3. Also $\tilde{P}=J_R'\tilde{P}'$ and therefore, Eq. (119), is transformed exactly to the same planar JPD $\tilde{P}(\tilde{\mathbf{R}},\tilde{\mathbf{S}})$ given by Eq. (101). This TPD is very similar, but not equal, to that calculated by Czopnik and Garbaczewski. The main difference between both probability densities is the expression of the determinant of matrix C, which accounts for the variance of those TPD's. In our proposal $\det C=(FG-H^2)^2$, which is exactly the same as that of the ordinary Brownian motion, and therefore the parameters F, G, and H do not contain the periodic functions $\sin\omega t$ and $\cos\omega t$ because they have been absorbed by the noise term $\tilde{\mathbf{A}}(t)$, as it is effectively corroborated in Eqs. (114)–(116). Furthermore, in Eq. (101), the factor $\omega H/\beta$ multiplies the cross product $(\tilde{\mathbf{S}}\times\tilde{\mathbf{R}})_z$. On the other hand, in Ref. [4], $\det C=(fg-h^2-k^2)^2$, where $g=G$, but the parameters f and h are not respectively equal to our parameters F and H due to the periodic functions appearing in f and h. The k parameter also contains such periodic functions, but it does not appear in our proposal because of the absorption of such periodic functions by the noise. Thus, k is an independent parameter in our proposal which multiplies the cross product $(\tilde{\mathbf{S}}\times\tilde{\mathbf{R}})_z$ appearing in the JPD of Ref. [4]. From the above arguments we can point out the following: when we make $\omega=0$, it can be checked that both TPD's reduce to the planar TPD of ordinary Brownian motion, as expected. In this case the parameters f, g, and h will be the same as those given in Eq. (40) together with $k=0$.

V. CONCLUSIONS

In this work we have shown that, with the change of variable $\mathbf{u}'=e^{-\mathbb{W}t}\mathbf{u}$, we can transform the Langevin equation (46) into a set of independent equations for $\mathbf{u}'$, as shown in Eq. (54). This last equation is practically the same as that of ordinary Brownian motion except by the term $\mathbb{R}^{-1}(t)\mathbf{A}(t)$, which is nothing but a rotation of the noise $\mathbf{A}(t)$. However, as we have shown, this rotating noise has the same statistical properties as that of $\mathbf{A}(t)$ and therefore such a rotation does not change the statistical properties of the original results, as can be seen by comparing the expressions (50) and (56). Because of this fact, the TPD for the velocity space $\mathbf{u}$ is easily calculated through the Fokker-Planck equation, as shown in Sec. III. Following the strategy of this last section, we calculate the TPD for the phase space $(\mathbf{r},\mathbf{u})$, which is decomposed into the product of two TPD's, one given by Eq. (101) which describes the diffusion process across the magnetic field, and that given by Eq. (102) which describes the diffusion process along the magnetic field. The planar TPD given in Eq. (101) has also been calculated by the method used by Czopnik and Garbaczewski, in order to compare with that calculated by these authors. Thus, our conclusion in this case is that both TPD's are similar, but not equal. Our proposal, as well as that in Ref. [4], are two different methodologies to describe the same physical problem.

Finally, our proposal can be extended to the study of the influence of a time-varying electric field and thus compared with the results of Refs. [5,6]. It also seems to be possible to apply our method to study the anisotropic diffusion across the magnetic field as studied in Ref. [7].

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